ELEN E3401: Electromagnetics
Spring 2025

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Lecture #9

Power Flow – lossless TL

Lossless TL: z = -d

$$\tilde{V}(d) = {V_0}^{+}(e^{j\beta d} + \Gamma e^{-j\beta d})$$

$$\tilde{I}(d) = \frac{{V_0}^{+}}{Z_0}(\underbrace{e^{j\beta d}}_{\text{incident}} \underbrace{-\Gamma e^{j\beta d}}_{\text{reflected}})$$

Time domain:

$$v(d,t) = |{V_0}^{+}|[\cos(\omega t + \beta d + \varphi^{+}) + |\Gamma|\cos(\omega t - \beta d + \varphi^{+} + \theta_r)]$$

$$i(d,t) = \frac{|{V_0}^{+}|}{Z_0}[\cos(\omega t + \beta d + \varphi^{+}) - |\Gamma|\cos(\omega t - \beta d + \varphi^{+} + \theta_r)]$$

Instantaneous Power: $P(d,t) = v(d,t)i(d,t)$

$$P(d,t) = \frac{|{V_0}^{+}|^2}{Z_0}[\underbrace{\cos^2(\omega t + \beta d + \varphi^{+})}_{(-d)\text{ direction toward load}} - \underbrace{|\Gamma|^2\cos^2(\omega t - \beta d + \varphi^{+} + \theta_r)}_{(+d)\text{ direction from load}}] \text{ [W]}$$

Power Flow – lossless TL

$$P(d,t) = \frac{|V_0{}^+|^2}{Z_0}\left[\cos^2(\omega t + \beta d + \varphi^+) - |\Gamma|^2 \cos^2(\omega t - \beta d + \varphi^+ + \theta_r)\right] \text{ [W]}$$

$$P^i(d,t) = \frac{|V_0{}^+|^2}{Z_0}\cos^2(\omega t + \beta d + \varphi^+) \text{ [W]}$$

$$P^r(d,t) = -|\Gamma|^2 \frac{|V_0{}^+|^2}{Z_0}\cos^2(\omega t - \beta d + \varphi^+ + \theta_r) \text{ [W]}$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

DC term

AC term

$$P^i(d,t) = \frac{|V_0{}^+|^2}{2Z_0}(1 + \cos(2\omega t + 2\beta d + 2\varphi^+)) \text{ [W]}$$ Incident power

$$P^r(d,t) = -|\Gamma|^2 \frac{|V_0{}^+|^2}{2Z_0}(1 + \cos(2\omega t - 2\beta d + 2\varphi^+ + 2\theta_r)) \text{ [W]}$$ Reflected power

Time-Average Power – lossless TL

Can obtain time-average power from time domain:

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$P_{avg}^{i}(d) = \frac{1}{T}\int_0^T P^i(d,t)dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P^i(d,t)dt$$

$$P_{avg}^{i} = \frac{|V_0{}^{+}|^2}{2Z_0} \text{ [W]} \qquad P_{avg}^{r} = -|\Gamma|^2\frac{|V_0{}^{+}|^2}{2Z_0} \text{ [W]} \quad \text{or} \quad P_{avg}^{r} = -|\Gamma|^2 P_{avg}^{i}$$

$$P_{avg} = P_{avg}^{i} + P_{avg}^{r} = \frac{|V_0{}^{+}|^2}{2Z_0}[1 - |\Gamma|^2]$$

Independent of *d*

(since TL is lossless)

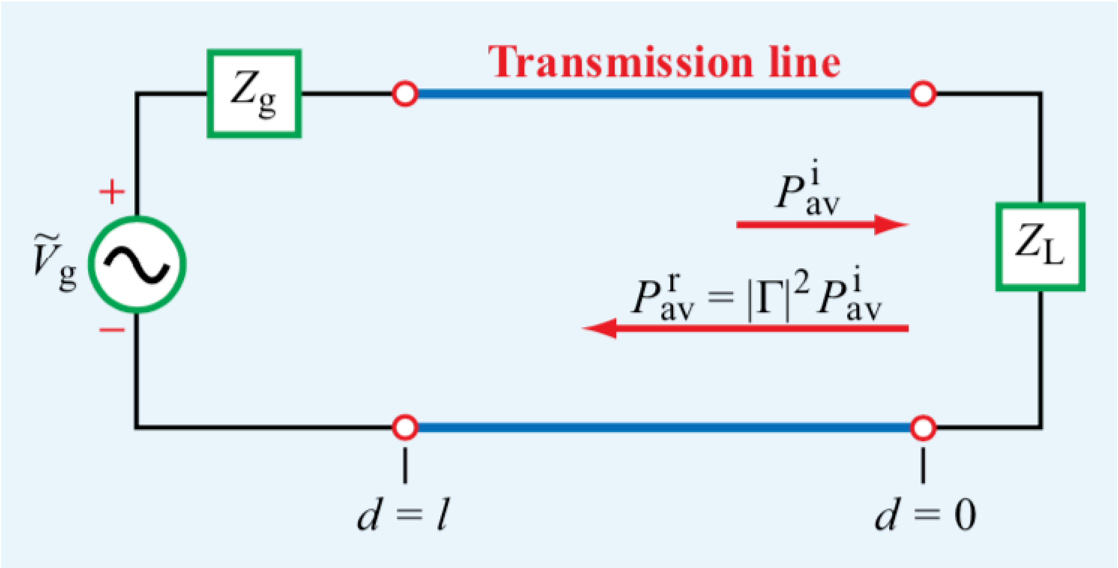
Time-Average Power – lossless TL

$$P_{avg}^{i} = \frac{|V_0^{+}|^2}{2Z_0}\ [\mathrm{W}] \qquad P_{avg}^{r} = -|\Gamma|^2 \frac{|V_0^{+}|^2}{2Z_0}\ [\mathrm{W}] \qquad P_{avg}^{r} = -|\Gamma|^2 P_{avg}^{i}$$

$$P_{avg} = P_{avg}^{i} + P_{avg}^{r} = \frac{|V_0^{+}|^2}{2Z_0}\left[1 - |\Gamma|^2\right]$$

<u>Independent of *d*</u>

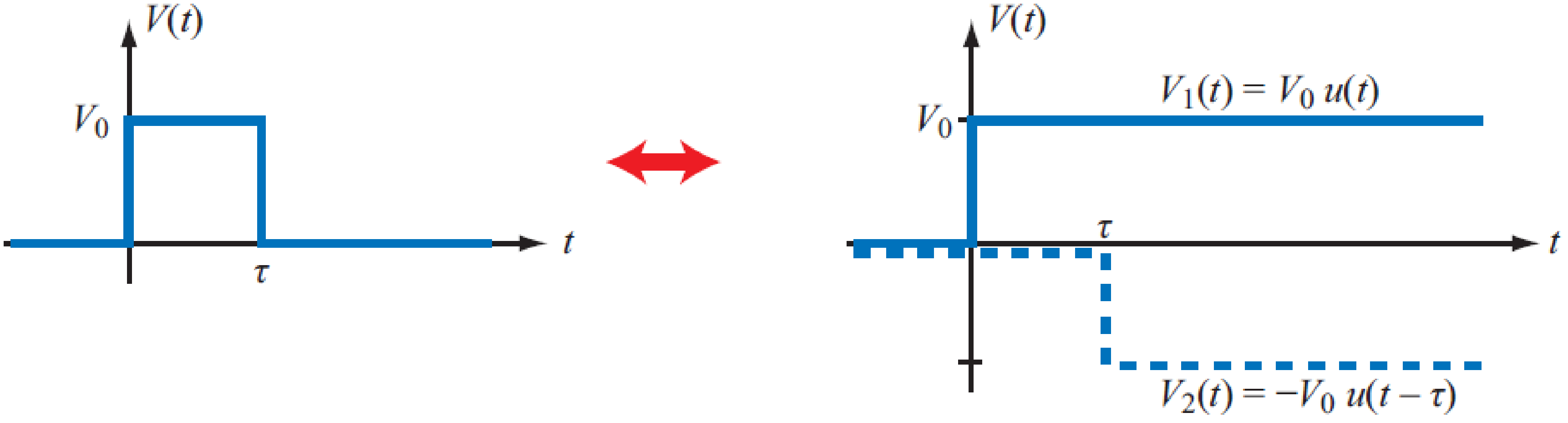
(since TL is lossless)



Phasor domain:

$$P_{avg} = \frac{1}{2} Re\{\tilde{V} \cdot \tilde{I}^{*}\}$$

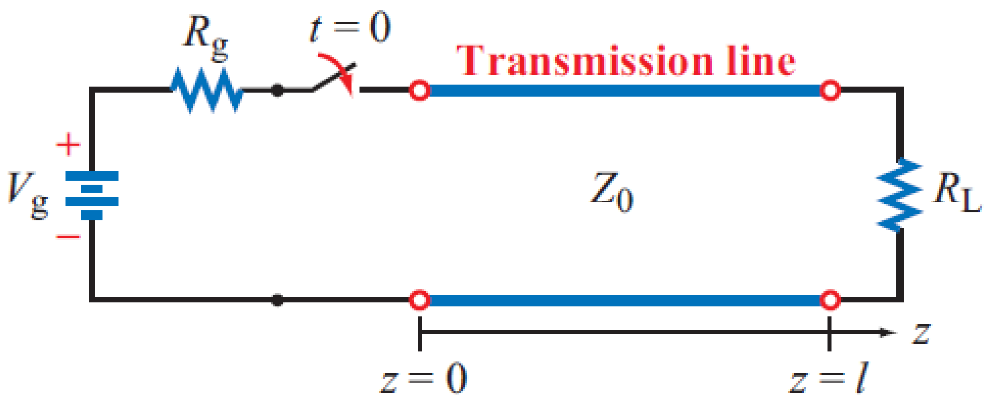
Transients



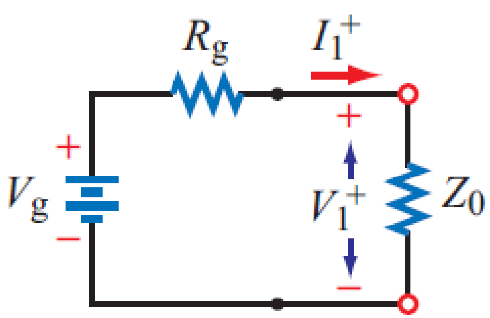
(a) Pulse of duration τ (b) $V(t) = V_1(t) + V_2(t)$

Rectangular pulse is equivalent to the sum of two step functions

Transient Response



(a) Transmission-line circuit



(b) Equivalent circuit at $t = 0^+$

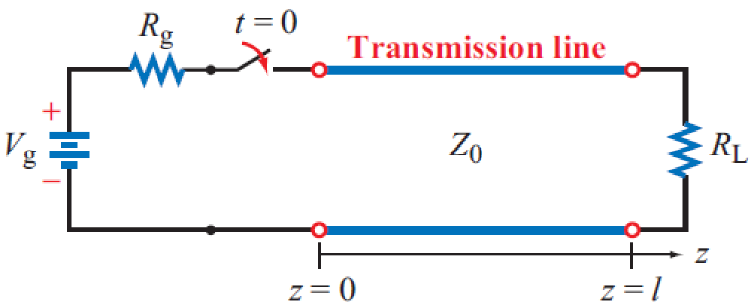
Initial conditions ($t=0^+$)

Zin = Zo (R_L does not appear)

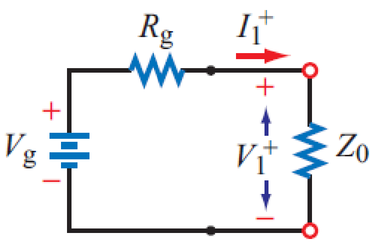
$$I_1^+ = \frac{V_g}{R_g + Z_0},$$

$$V_1^+ = I_1^+ Z_0 = \frac{V_g Z_0}{R_g + Z_0}$$

Transient Response



(a) Transmission-line circuit



(b) Equivalent circuit at $t = 0^+$

Initial current and voltage

$$I_1^+ = \frac{V_g}{R_g + Z_0},$$

$$V_1^+ = I_1^+ Z_0 = \frac{V_g Z_0}{R_g + Z_0}$$

Reflection at the load

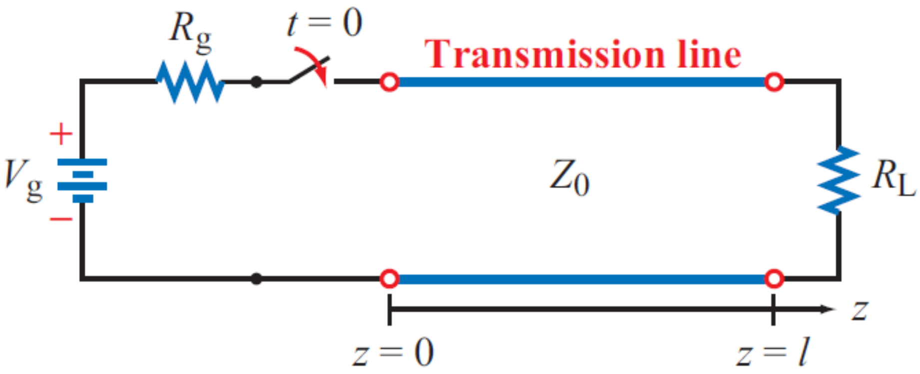
$$V_1^- = \Gamma_L V_1^+,$$

Load reflection coefficient $$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

Second transient $$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$$

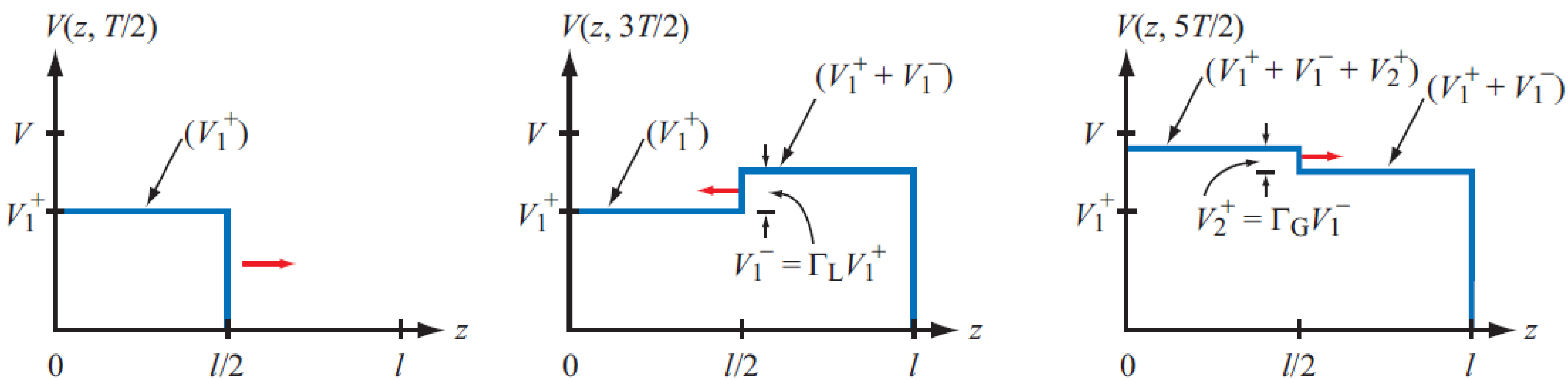
Generator reflection coefficient $$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}$$

Voltage Wave



(a) Transmission-line circuit

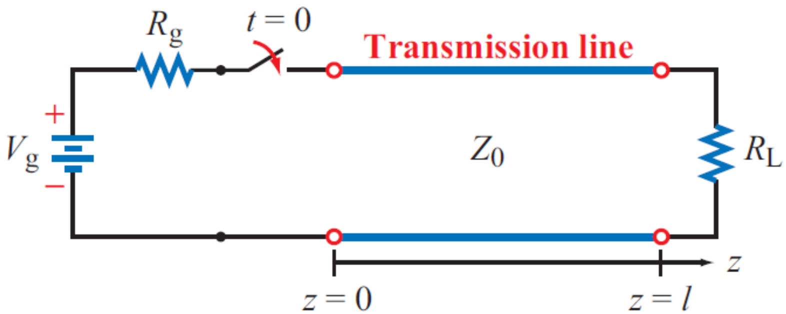
$T = l/u_p$ is the time it takes the wave to travel the full length of the line



(a) $V(z)$ at $t = T/2$ (b) $V(z)$ at $t = 3T/2$ (c) $V(z)$ at $t = 5T/2$

$R_g = 4Z_0$ and $R_L = 2Z_0$. The corresponding reflection coefficients are $\Gamma_L = 1/3$ and $\Gamma_g = 3/5$.

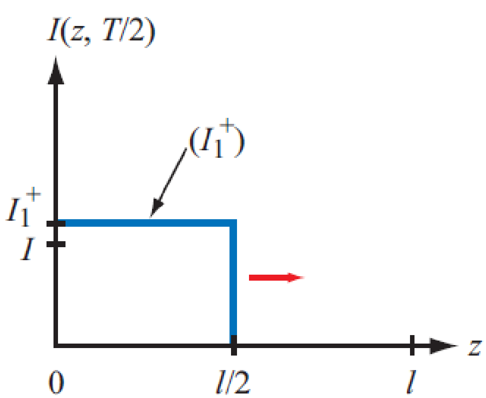
Current Wave



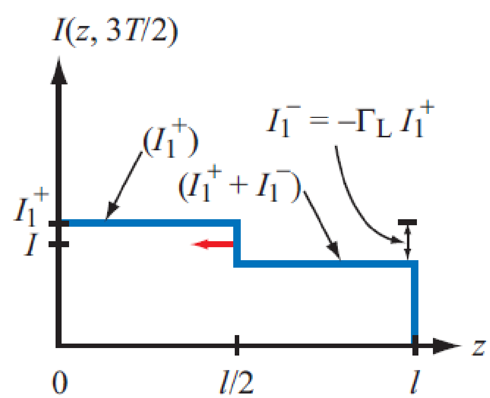
(a) Transmission-line circuit

Reflection coefficient for current is the negative of that for voltage

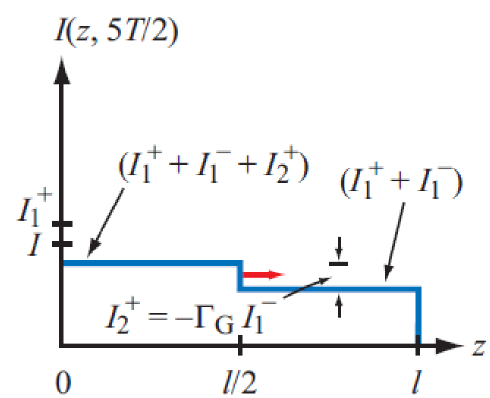
$$I_1^- = -\Gamma_L I_1^+,$$
$$I_2^+ = -\Gamma_g I_1^- = \Gamma_g \Gamma_L I_1^+$$



(d) $I(z)$ at $t = T/2$

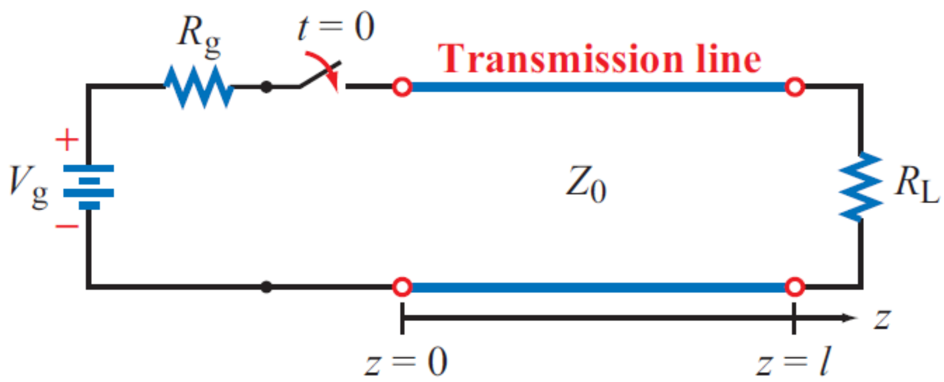


(e) $I(z)$ at $t = 3T/2$



(f) $I(z)$ at $t = 5T/2$

Steady State Response



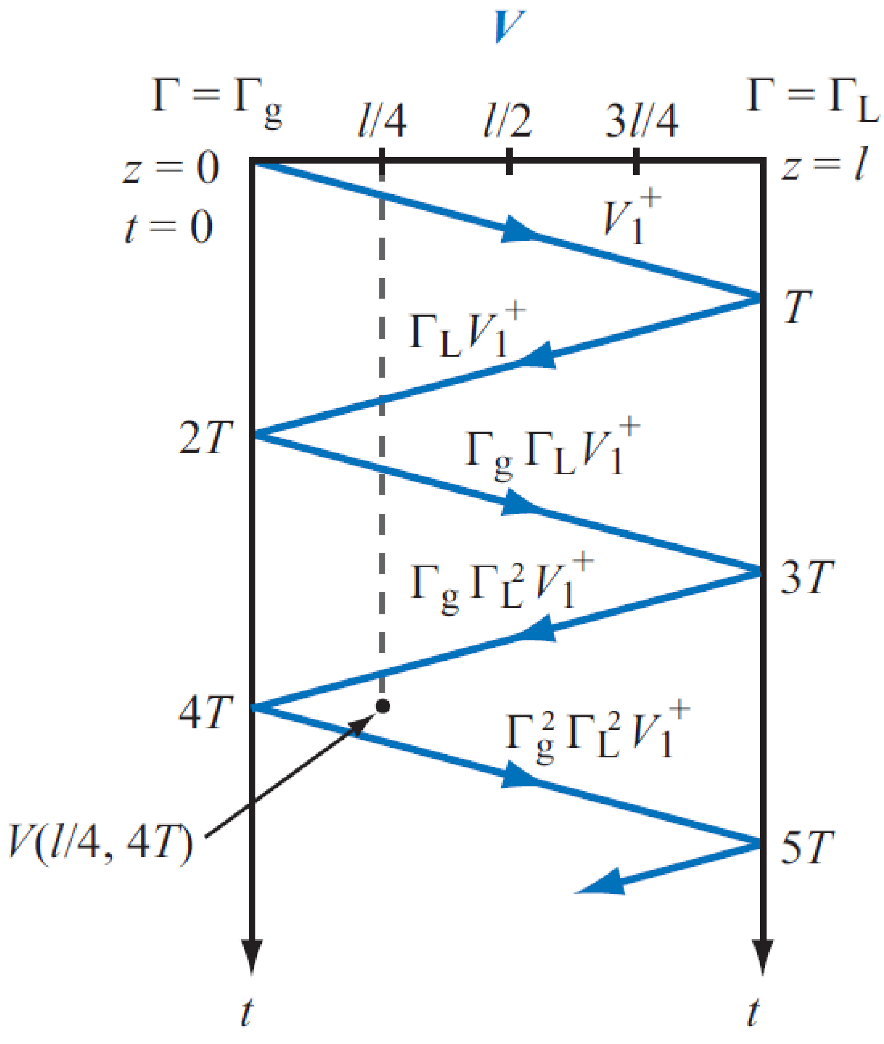
(a) Transmission-line circuit

$$V_\infty = \frac{V_g R_L}{R_g + R_L}$$

$$I_\infty = \frac{V_\infty}{R_L} = \frac{V_g}{R_g + R_L}$$

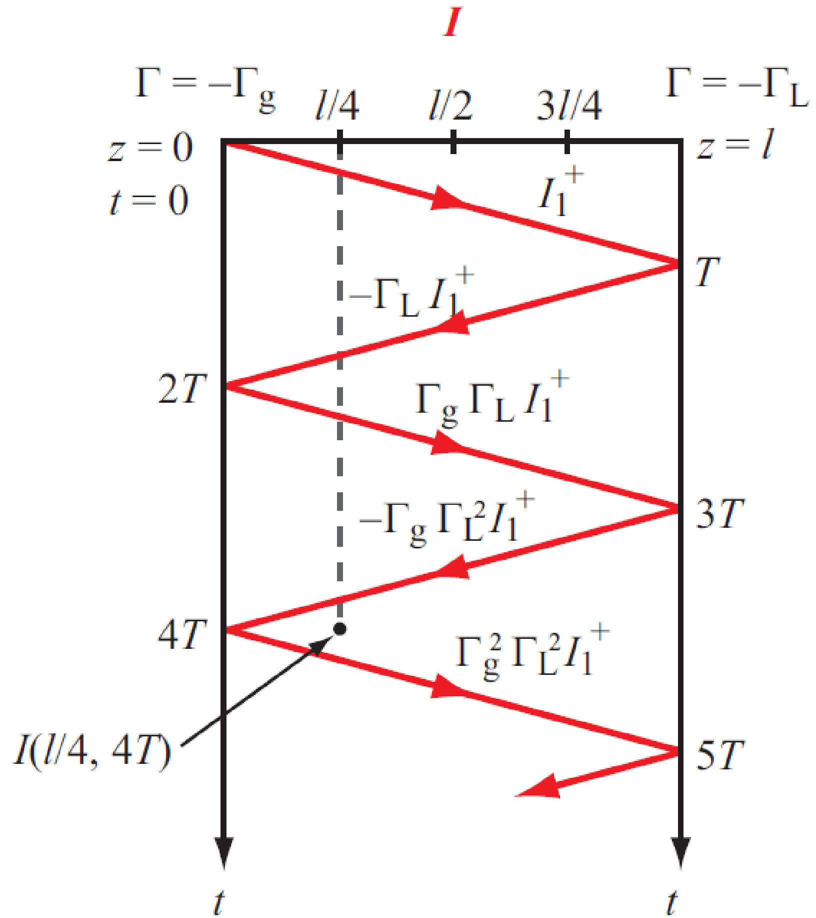
Bounce Diagram

Voltage



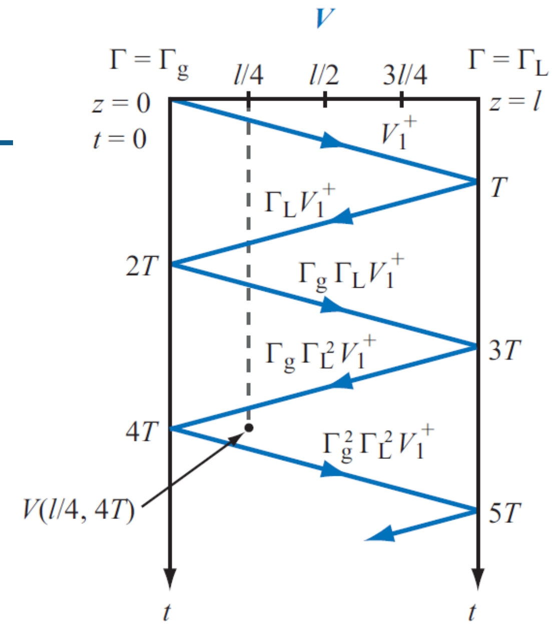
(a) Voltage bounce diagram

Bounce Diagram
Current

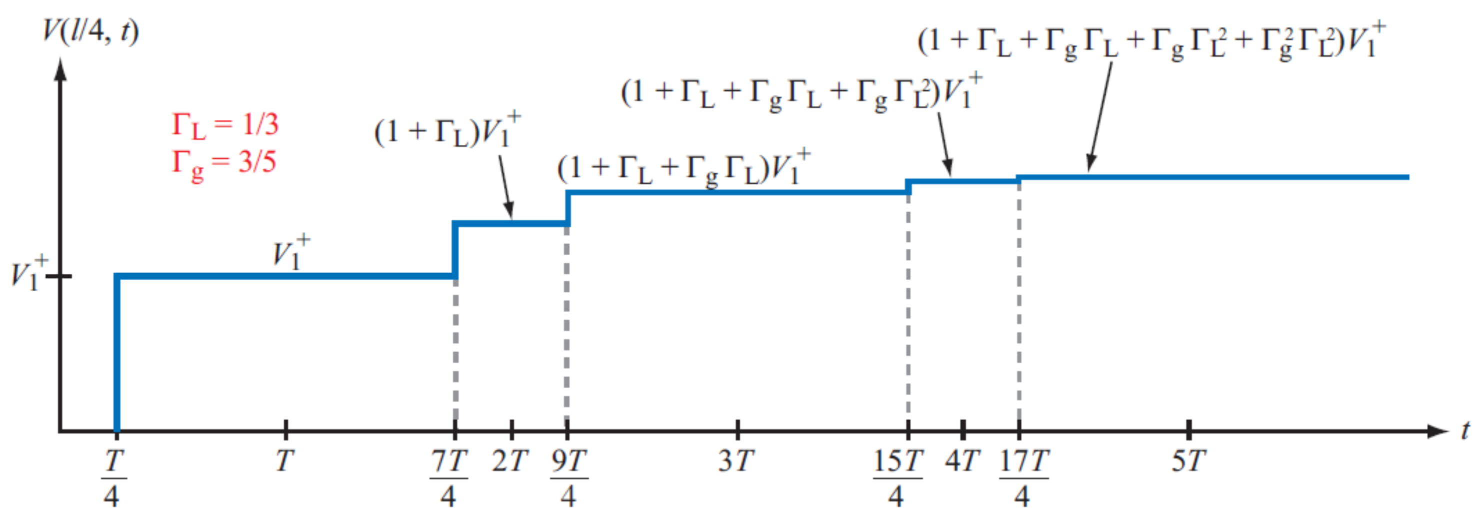


(b) Current bounce diagram

Voltage as function of time at z=l/4

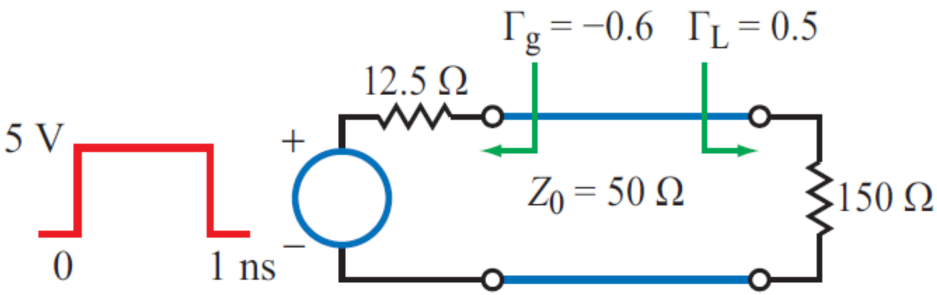


(a) Voltage bounce diagram

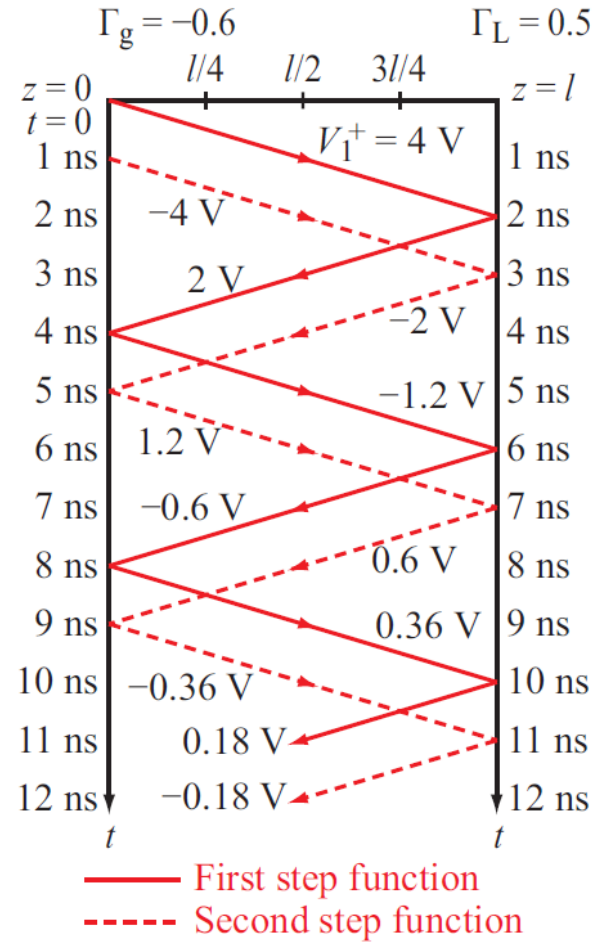


(c) Voltage versus time at $z = l/4$

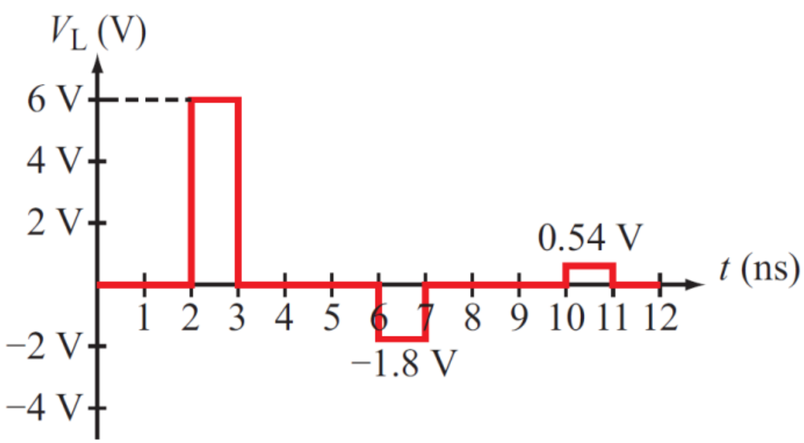
Example



(a) Pulse circuit



(b) Bounce diagram



(c) Voltage waveform at the load

Summary

Chapter 2 Relationships

TEM Transmission Lines

$$L'C' = \mu\varepsilon$$

$$\frac{G'}{C'} = \frac{\sigma}{\varepsilon}$$

$$\alpha = \mathfrak{Re}(\gamma) = \mathfrak{Re}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(Np/m)}$$

$$\beta = \mathfrak{Im}(\gamma) = \mathfrak{Im}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(rad/m)}$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad (\Omega)$$

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

Step Function Transient Response

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0}$$

$$V_\infty = \frac{V_g R_L}{R_g + R_L}$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

Lossless Line

$$\alpha = 0$$

$$\beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(m/s)}$$

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\,\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

$$d_{max} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2}$$

$$d_{min} = \frac{\theta_r \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}\,[1 - |\Gamma|^2]$$

Vector fields analysis (review)

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\|\mathbf{A}\| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A} \cdot \mathbf{B} =$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

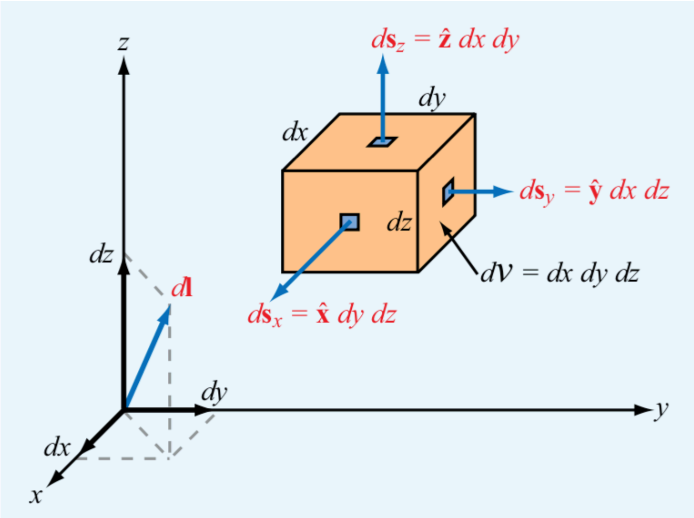
Cartesian Coordinates

Differential length vector: $d\vec{l} = \hat{x}dl_x + \hat{y}dl_y + \hat{z}dl_z = \hat{x}\mathrm{dx} + \hat{y}dy + \hat{z}dz$

Differential area vector: $d\vec{S}_x = \hat{x}dl_y dl_z = \hat{x}dydz$ (y-z plane)

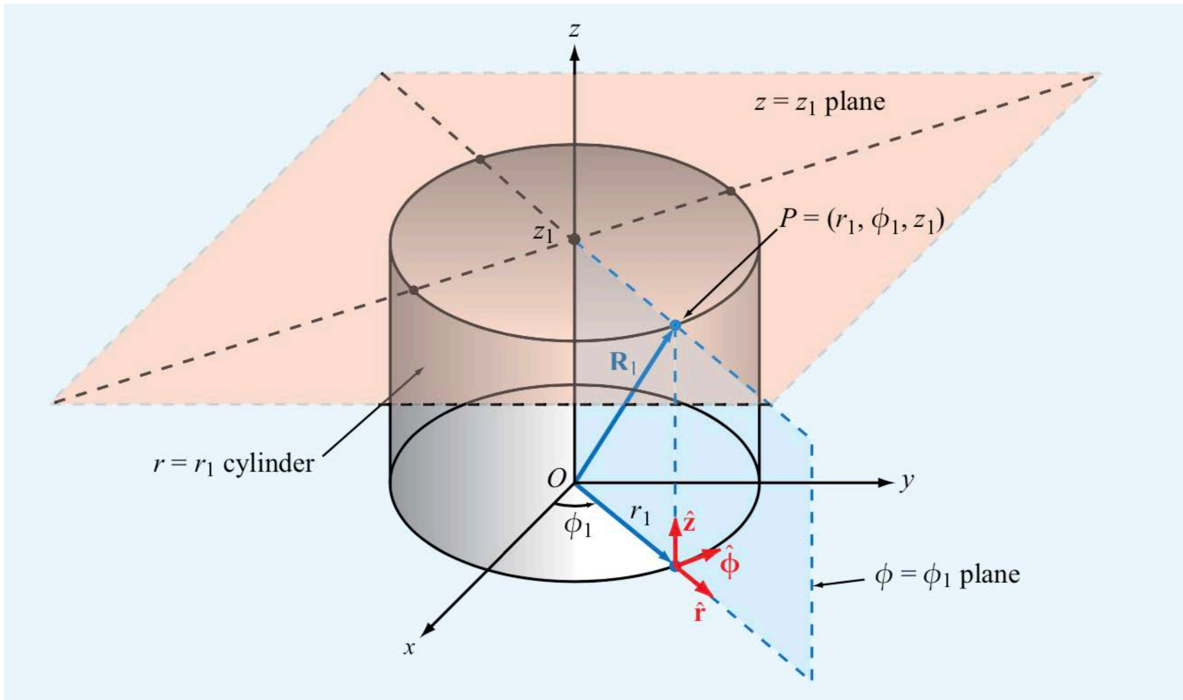
$$d\vec{S}_y = \hat{y}dxdz$$

$$d\vec{S}_z = \hat{z}dxdy$$



Differential volume: $d\mathcal{V} = dxdydz$

Cylindrical Coordinates

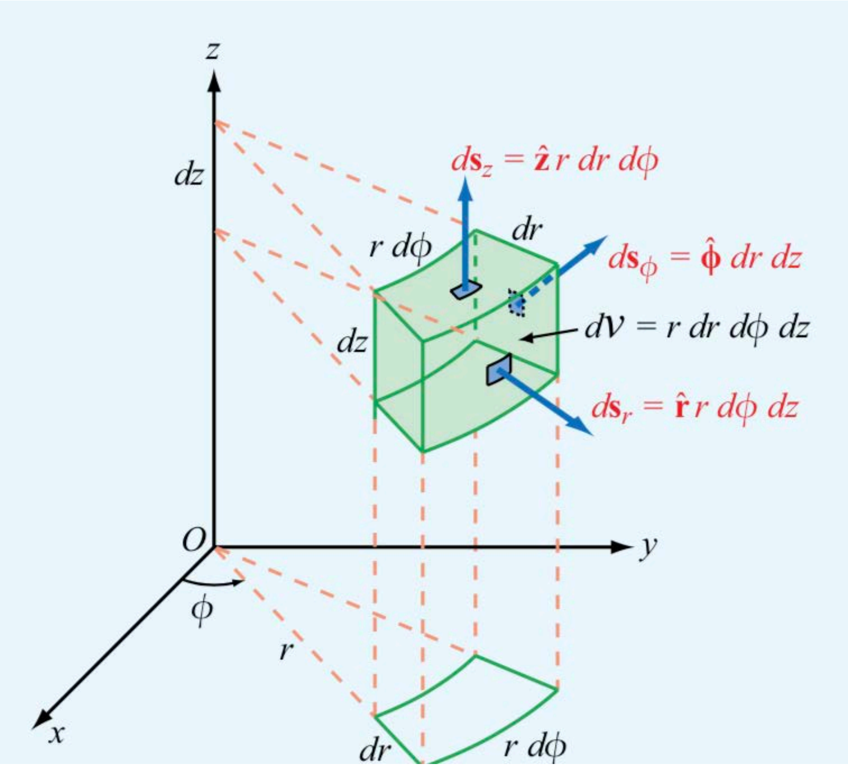


$0 \leq r < \infty$ $\qquad$ $0 \leq \varphi < 2\pi$ $\qquad$ $-\infty < z < \infty$

$\hat{r} \times \hat{\varphi} = \hat{z}$ $\quad$ $\hat{\varphi} \times \hat{z} = \hat{r}$ $\quad$ $\hat{z} \times \hat{r} = \hat{\varphi}$ $\qquad$ r → φ → z

$$\vec{A} = \hat{a}\left|\vec{A}\right| = \hat{r}A_r + \hat{\varphi}A_\varphi + \hat{z}A_z$$

Cylindrical Coordinates



Differential Length:

$$dl_r = dr, dl_\varphi = rd\varphi, dl_z = dz$$

$$d\vec{l} = \hat{r}dl_r + \hat{\varphi}dl_\varphi + \hat{z}dl_z$$

$$d\vec{l} = \hat{r}dr + \hat{\varphi}rd\varphi + \hat{z}dz$$

Differential surface area:

$$d\vec{S}_r = \hat{r}dl_\varphi dl_z = \hat{r}rd\varphi dz$$ ($\varphi - z$ cylindrical surface)

$$d\vec{S}_\varphi = \hat{\varphi}dl_r dl_z = \hat{\varphi}drdz$$ ($r - z$ plane)

$$d\vec{S}_z = \hat{z}dl_r dl_\varphi = \hat{z}rdrd\varphi$$ ($r - \varphi$ plane)

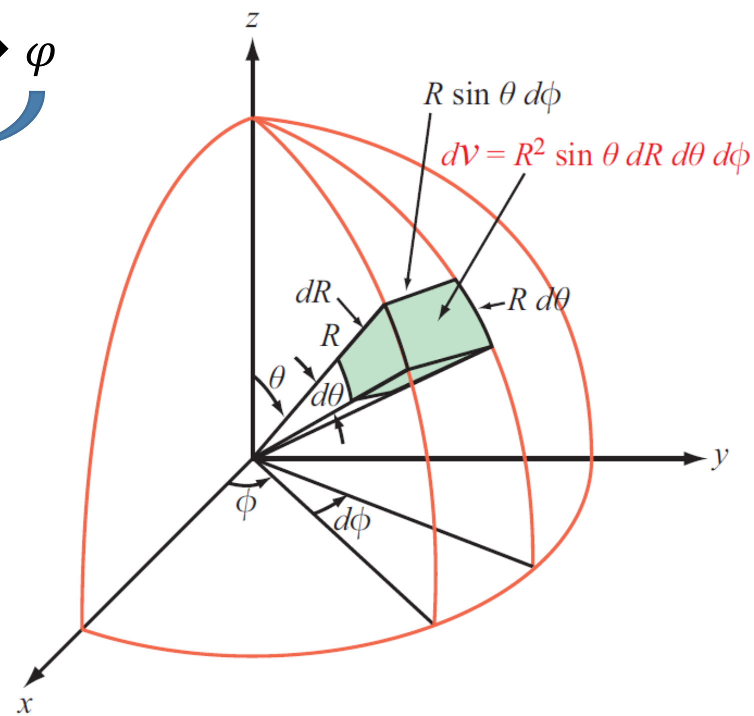
Differential volume: $d\mathcal{V} = dl_r dl_\varphi dl_z = r\, drd\varphi dz$

Spherical Coordinates

$\hat{R} \times \hat{\theta} = \hat{\varphi}$ $\quad \hat{\theta} \times \hat{\varphi} = \hat{R}$ $\quad \hat{\varphi} \times \hat{R} = \hat{\theta}$ $\quad R \rightarrow \theta \rightarrow \varphi$

$\vec{A} = \hat{a}|\vec{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi$

$0 \leq R < \infty \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \varphi < 2\pi$



Differential Length:

$$dl_R = dR, dl_\theta = Rd\theta, dl_\varphi = Rsin\theta d\varphi$$

$$d\vec{l} = \hat{R}dl_R + \hat{\theta}dl_\theta + \hat{\varphi}dl_\varphi$$

$$d\vec{l} = \hat{R}dR + \hat{\theta}Rd\theta + \hat{\varphi}Rsin\theta d\varphi$$

Differential surface area:

$$d\vec{S}_R = \hat{R}dl_\theta dl_\varphi = \hat{R}R^2 sin\theta d\theta d\varphi$$ ($\theta - \varphi$ spherical surface)

$$d\vec{S}_\theta = \hat{\theta}dl_R dl_\varphi = \hat{\theta}Rsin\theta dRd\varphi$$ (R $- \varphi$ plane)

$$d\vec{S}_\varphi = \hat{\varphi}dl_R dl_\theta = \hat{\varphi}RdRd\theta$$ ($R - \theta$ plane)

Differential volume: $\quad d\mathcal{V} = dl_R dl_\theta dl_\varphi = R^2 sin\theta dRd\theta d\varphi$

Transformations

<u>Cartesian to cylindrical:</u>

$$r = \sqrt[+]{x^2 + y^2} \qquad \varphi = tan^{-1}\left(\frac{y}{x}\right)$$

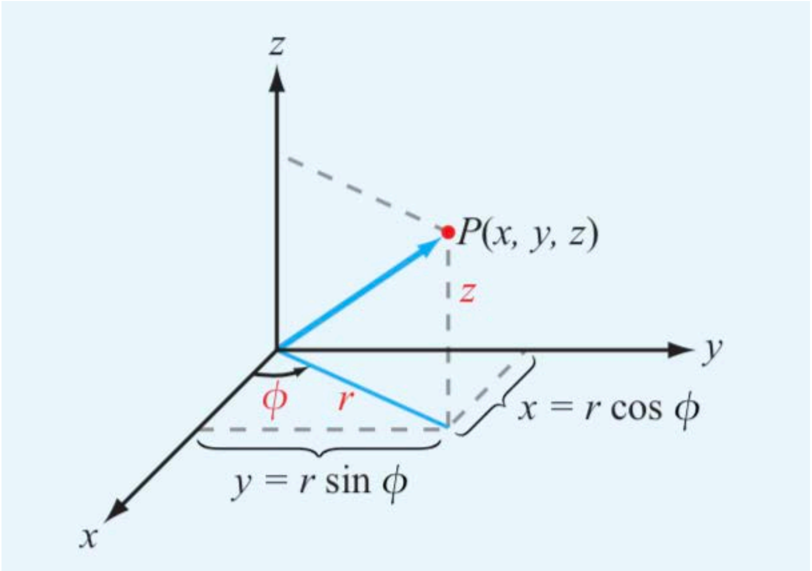
$$x = rcos\varphi \quad y = rsin\varphi$$

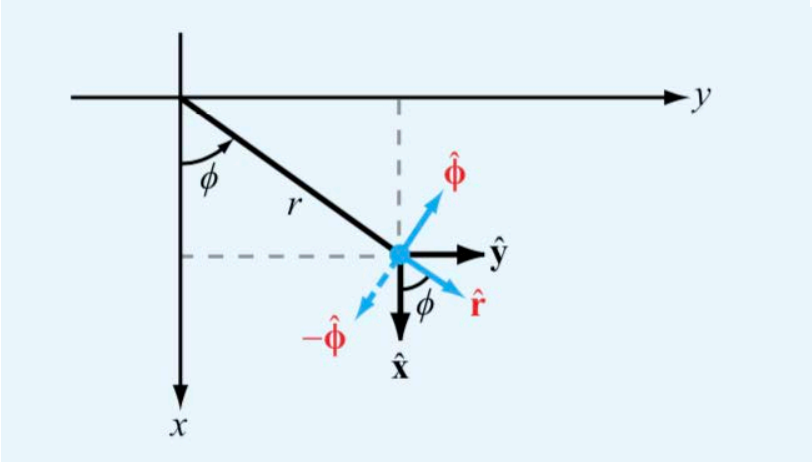
$$\hat{r} \cdot \hat{x} = cos\varphi \quad \hat{r} \cdot \hat{y} = sin\varphi$$

$$\hat{\varphi} \cdot \hat{x} = -sin\varphi \quad \hat{\varphi} \cdot \hat{y} = cos\varphi$$

$$\begin{cases} \hat{r} = \hat{x}cos\varphi + \hat{y}sin\varphi \\ \hat{\varphi} = -\hat{x}sin\varphi + \hat{y}cos\varphi \end{cases}$$

$$\begin{cases} \hat{x} = \hat{r}cos\varphi - \hat{\varphi}sin\varphi \\ \hat{y} = \hat{r}sin\varphi + \hat{\varphi}cos\varphi \end{cases}$$





Transformations

<u>Cartesian to spherical:</u>

$$R = {}^{+}\sqrt{x^2 + y^2 + z^2} \qquad \theta = tan^{-1}\left(\frac{{}^{+}\sqrt{x^2 + y^2}}{z}\right) \qquad \varphi = tan^{-1}\left(\frac{y}{x}\right)$$

$$x = Rsin\theta cos\varphi \qquad y = Rsin\theta sin\varphi \qquad z = Rcos\theta$$

$$\begin{cases} \hat{R} = \hat{x}sin\theta cos\varphi + \hat{y}sin\theta sin\varphi + \hat{z}cos\theta \\ \hat{\theta} = \hat{x}cos\theta cos\varphi + \hat{y}cos\theta sin\varphi - \hat{z}sin\theta \\ \hat{\varphi} = -\hat{x}sin\varphi + \hat{y}cos\varphi \end{cases}$$

$$\begin{cases} \hat{x} = \hat{R}sin\theta cos\varphi + \hat{\theta}cos\theta cos\varphi - \hat{\varphi}sin\varphi \\ \hat{y} = \hat{R}sin\theta sin\varphi + \hat{\theta}cos\theta sin\varphi + \hat{\varphi}cos\varphi \\ \hat{z} = \hat{R}cos\theta - \hat{\theta}sin\theta \end{cases}$$

Gradient of Scalar Field

We will introduce: Gradient → for scalar fields

Divergence, curl → for vector fields

Example of gradient → scalar field → temperature

Vector Analysis

$\vec{E}$ and $\vec{H}$ are vector fields

Scalar Field → temperature $T(x, y, z)$

$$d\vec{l} = \hat{x}\mathrm{dx} + \hat{y}dy + \hat{z}dz$$

dT along differential direction $d\vec{l}$

$$dT = \underbrace{\left[\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right]}_{\text{Vector = temperature change in direction}} \cdot d\vec{l}$$

Gradient of T (*grad T*)

$$\vec{\nabla}T = grad\, T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

Define gradient operator: $\boxed{\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$

Gradient of Scalar Field

Gradient operator on scalar function → results in vector with:

mag = max rate of change of physical scalar per unit distance

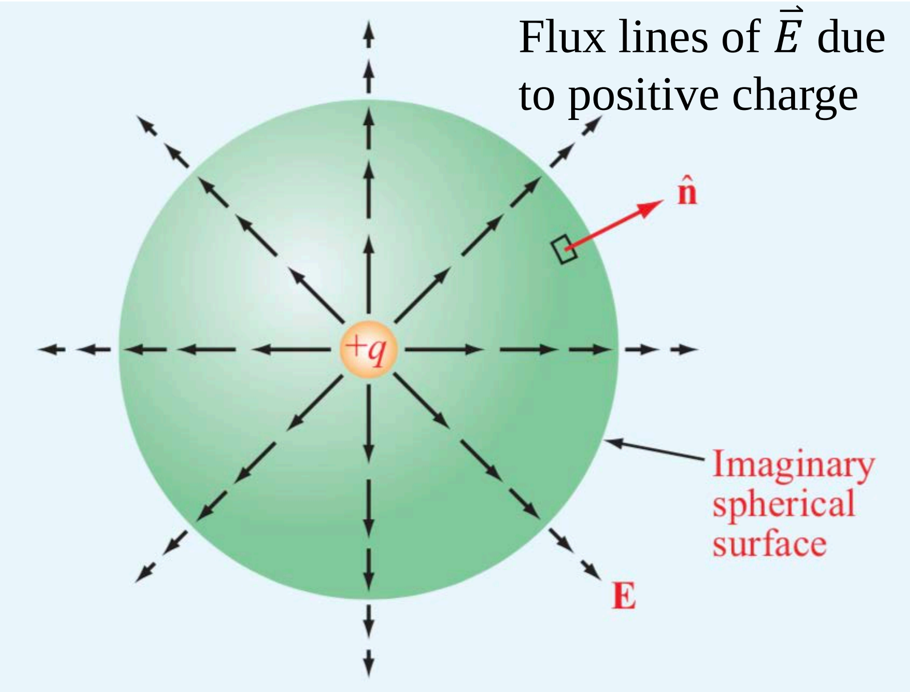
Direction = direction of max increase

$$\boxed{\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

$$\vec{\nabla}_{cyl} = \hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}_{sph} = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{Rsin\theta}\frac{\partial}{\partial \varphi}$$

Divergence of Vector Field



Recall Coulomb's Law → positive point charge q

$\vec{E}$ field will point outward, proportional to q and decreases as $\frac{1}{R^2}$

$$\vec{E} = \hat{R}\frac{q}{4\pi\epsilon_0 R^2} \text{ (V/m)}$$

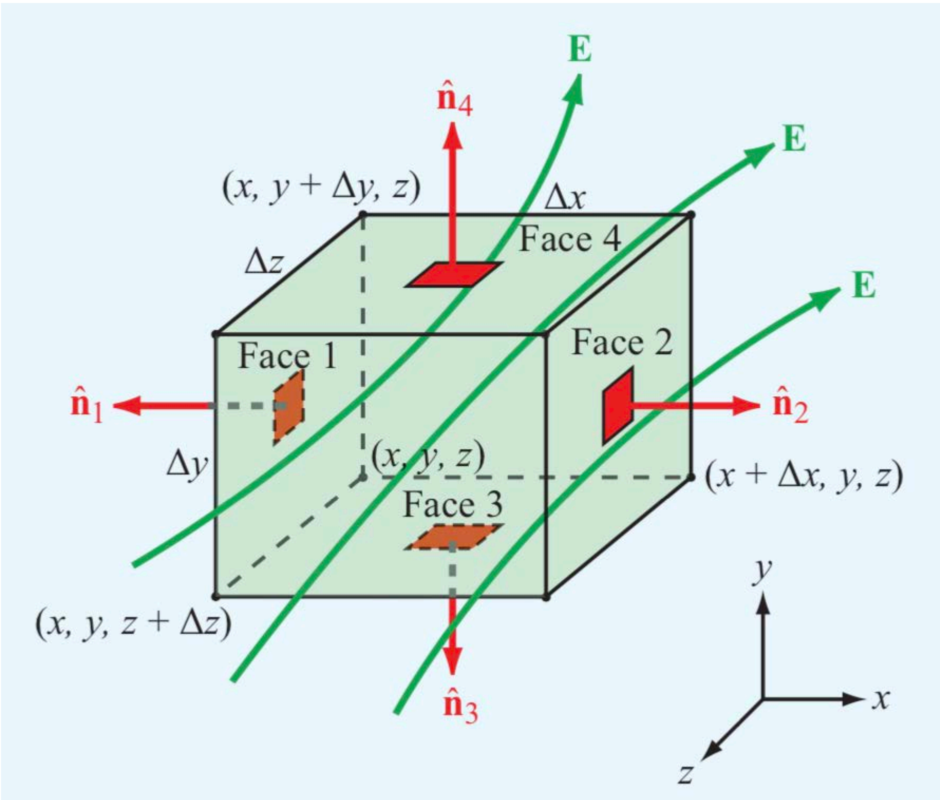
At surface boundary → define <u>flux density</u>: amount of outward flux crossing unit surface

<u>Flux density</u> of $\vec{E} = \frac{\vec{E}\cdot d\vec{s}}{|d\vec{s}|} = \frac{\vec{E}\cdot\hat{n}ds}{ds} = \vec{E}\cdot\hat{n}$, $\hat{n}$ = normal to $d\vec{s}$

Total flux = $\oint_S \vec{E}\cdot d\vec{s}$

Enclosed imaginary surface

Divergence of Vector Field



Differential rectangular volume for divergence of $\vec{E}$

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

We sum up the flux through the 6 faces:

Face 1

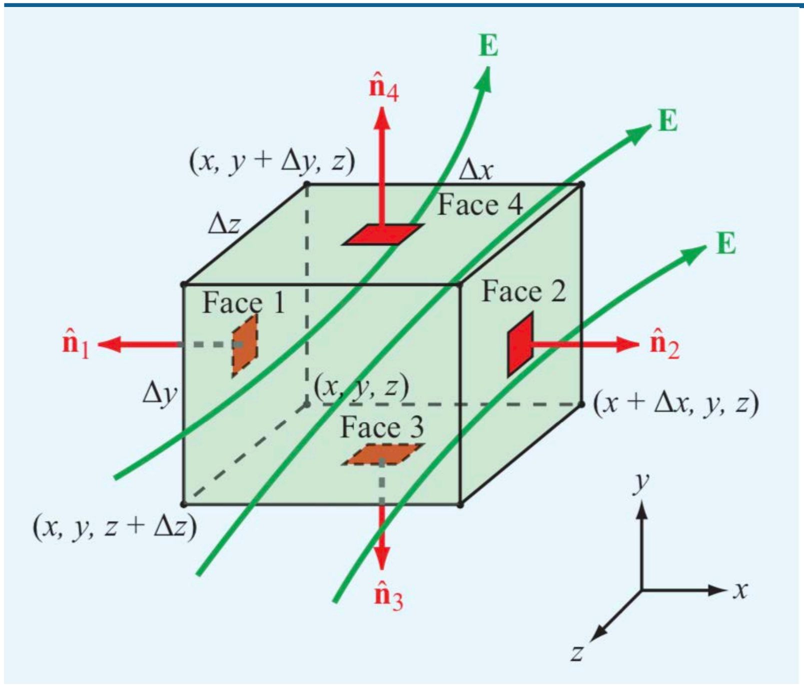
$$F_1 = \int_{face1} \vec{E} \cdot \hat{n}_1 ds$$

$$= \int (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \cdot (-\hat{x}) dydz$$

$$F_1 = -E_x(1)\Delta y \Delta z$$

Value at center (small surface)

Divergence of Vector Field



Differential rectangular volume for divergence of $\vec{E}$

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

<u>Face 2</u>

$$F_2 = E_x(2)\Delta y \Delta z$$

We shrink volume to Δx and have:

$$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x}\Delta \mathrm{x} \quad \text{(Taylor series 1}^{\text{st}}\text{ order)}$$

$$F_2 = [E_x(1) + \frac{\partial E_x}{\partial x}\Delta \mathrm{x}]\Delta y \Delta z$$

$$F_1 + F_2 = \frac{\partial E_x}{\partial x}\Delta \mathrm{x}\Delta y \Delta z$$

Do same $\underbrace{F_3, F_4}_{\frac{\partial E_y}{\partial y}}, \underbrace{F_5, F_6}_{\frac{\partial E_z}{\partial z}}$

Divergence of Vector Field

$$\oint_S \vec{E} \cdot d\vec{s} = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] \Delta x \Delta y \Delta z$$

$$\oint_S \vec{E} \cdot d\vec{s} = (div\ \vec{E})\Delta \mathcal{V}$$ ← Differential volume

Shrink $\Delta \mathcal{V} \rightarrow 0$ div $\vec{E}$: divergence of $\vec{E}$ → div $\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

Shrink $\Delta \mathcal{V} \rightarrow 0$ → divergence of $\vec{E}$ defined at a point as (div $\vec{E}$): net outward flux per unit volume over closed surface:

$$\text{div } \vec{E} \equiv \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{E} \cdot d\vec{s}}{\Delta \mathcal{V}}$$

S is surface encloses elemental volume ***V***

We use $\vec{\nabla} \cdot \vec{E}$ to indicate div $\vec{E}$

$$\boxed{\vec{\nabla} \cdot \vec{E} = div\ \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

$\vec{\nabla} \cdot \vec{E}$:
- positive if net flux lines out so enclosed volume contains source
- negative if net flux lines in so enclosed volume contains sink
- Zero: net flux =0 → divergence-less

Divergence of Vector Field

$$\oint_S \vec{E} \cdot d\vec{s} = \vec{\nabla} \cdot \vec{E} \Delta \mathcal{V} \rightarrow \int_v \vec{\nabla} \cdot \vec{E} \, d\mathcal{V}$$

Divergence theorem

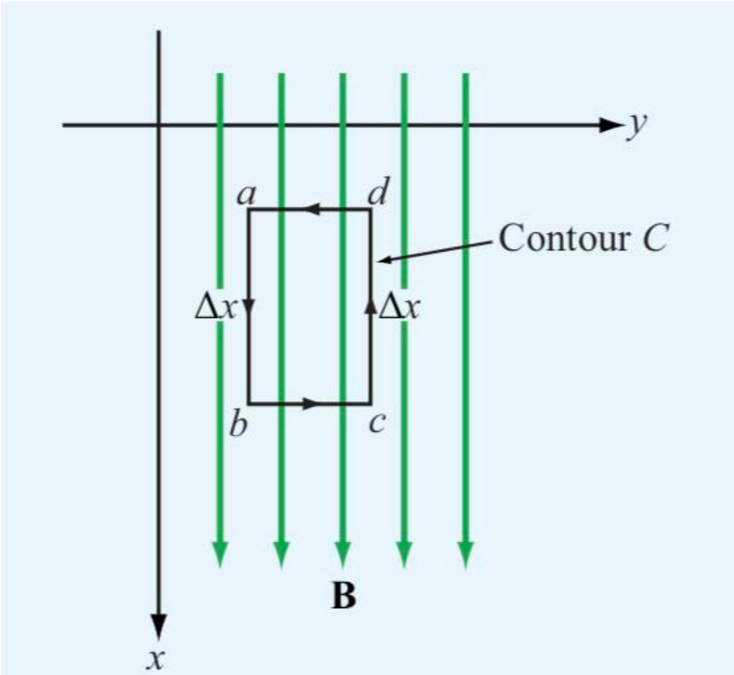
$$\underbrace{\int_v \vec{\nabla} \cdot \vec{E} \, d\mathcal{V}}_{\text{Divergence of field from volume}} = \underbrace{\oint_S \vec{E} \cdot d\vec{s}}_{\text{Sum of flux through surface enclosing the volume}}$$

Curl of Vector Field

Curl of vector, $\vec{B}$ → rotation or circulation

$$circulation = \oint_C \vec{B} \cdot d\vec{l}$$

Consider uniform $\vec{B}$ field: $\vec{B} = \hat{x}B_o$ magnetic flux density

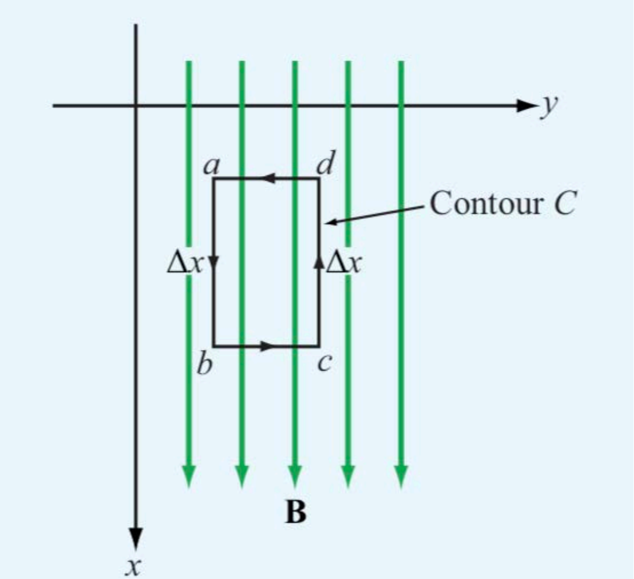


(a) Uniform field

Curl of Vector Field

Curl of vector, $\vec{B}$ → rotation or circulation: $circulation = \oint_C \vec{B} \cdot d\vec{l}$

Uniform $\vec{B}$ field: $\vec{B} = \hat{x}B_o$ magnetic flux density



(a) Uniform field

$$circulation = \int_a^b \hat{x}B_o \cdot \hat{x}dx + \underbrace{\int_b^c \hat{x}B_o \cdot \hat{y}dy}_{= 0 \text{ since } \hat{x}\cdot\hat{y} = 0} + \int_c^d \hat{x}B_o \cdot \hat{x}dx + \underbrace{\int_d^a \hat{x}B_o \cdot \hat{y}dy}_{= 0 \text{ since } \hat{x}\cdot\hat{y} = 0}$$

$$circulation = B_o\Delta \mathrm{x} - B_o\Delta \mathrm{x} = 0$$

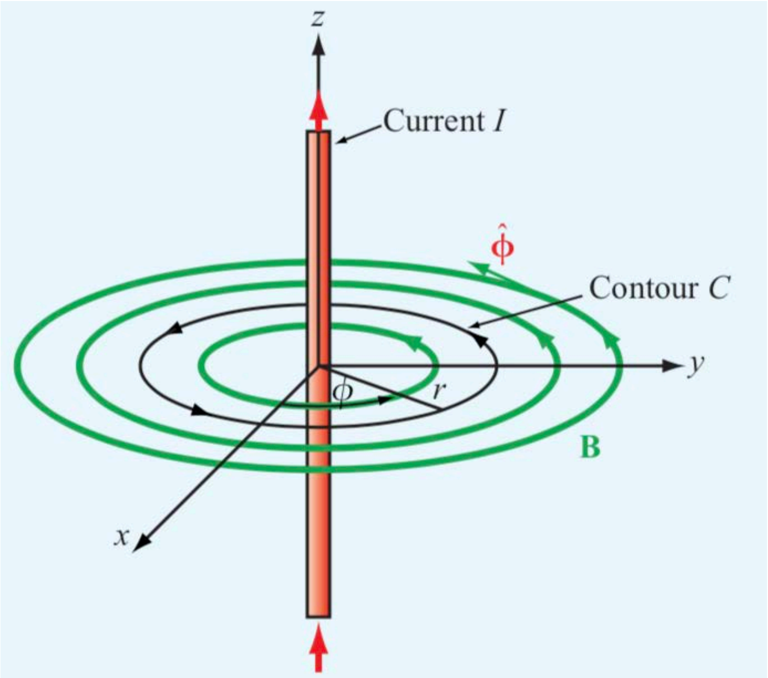
$$\Delta x = b - a = c - d$$

Circulation of <u>uniform</u> field = 0

Curl of Vector Field

Consider magnetic flux density, $\vec{B}$ of infinite wire with dc-current = I

$$\vec{B} = \hat{\varphi}\frac{\mu_0 I}{2\pi r}$$



(b) Azimuthal field

For circular contour, C, at radius r in x-y plane: $d\vec{l} = \hat{\varphi} r d\varphi$

$$circulation\ of\ \vec{B} = \int_c \vec{B} \cdot d\vec{l}$$

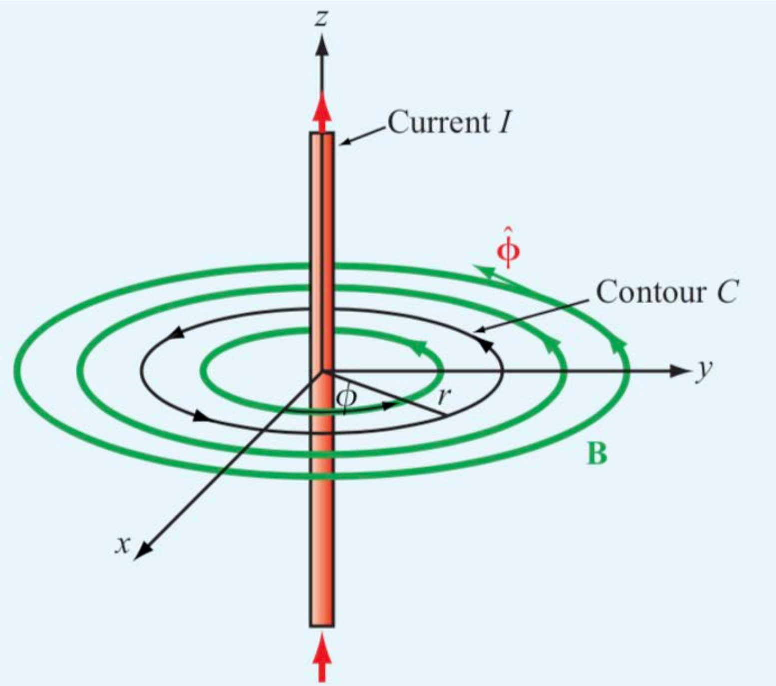
$$circulation\ = \int_0^{2\pi} \hat{\varphi}\frac{\mu_0 I}{2\pi r} \cdot \hat{\varphi} r d\varphi = \mu_0 I$$

Curl of Vector Field

magnetic flux density, $\vec{B}$ of infinite wire with dc-current = I

$$\vec{B} = \hat{\varphi}\,\frac{\mu_0 I}{2\pi r}$$

$$circulation\ of\ \vec{B} = \int_c \vec{B}\cdot d\vec{l}$$



(b) Azimuthal field

Take contour in x-z or y-z planes

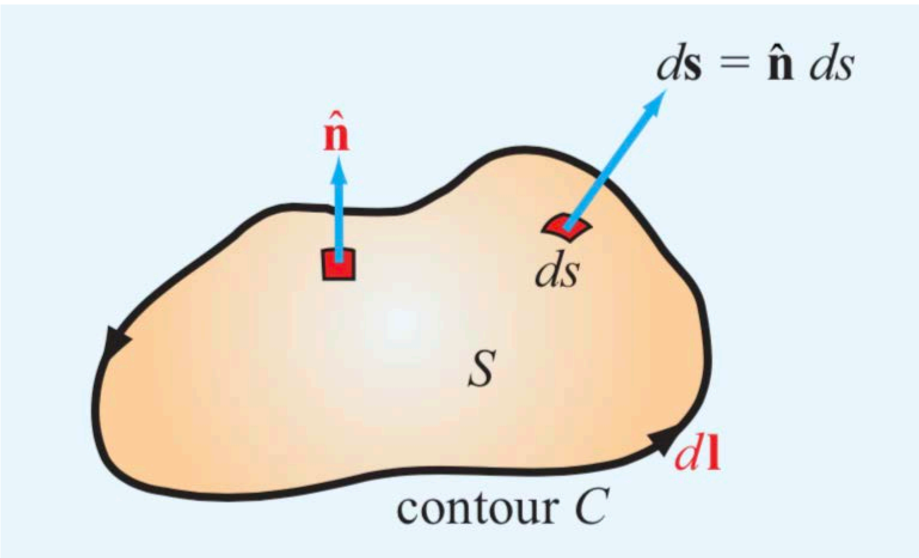
<u>circulation = 0</u> because there is <u>no φ component</u>

→ Circulation depends on <u>contour</u> and <u>direction</u>

$$\vec{\nabla}\times\vec{B} = \lim_{\Delta s\to 0}\frac{1}{\Delta s}\left[\hat{n}\oint_c \vec{B}\cdot d\vec{l}\right]_{max}$$

Curl of Vector Field

$\vec{\nabla} \times \vec{B}$ = curl of $\vec{B}$ = circulation of $\vec{B}$ per unit area, with area ΔS of contour C selected such that circulation is maximum



$$\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$$

$$\vec{\nabla} \times \vec{B} = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$$

$$+\hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Useful Vector Identities

$$\vec{\nabla} \times (\vec{A} + \vec{B}) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} V) = 0$$

Stoke’s Theorem

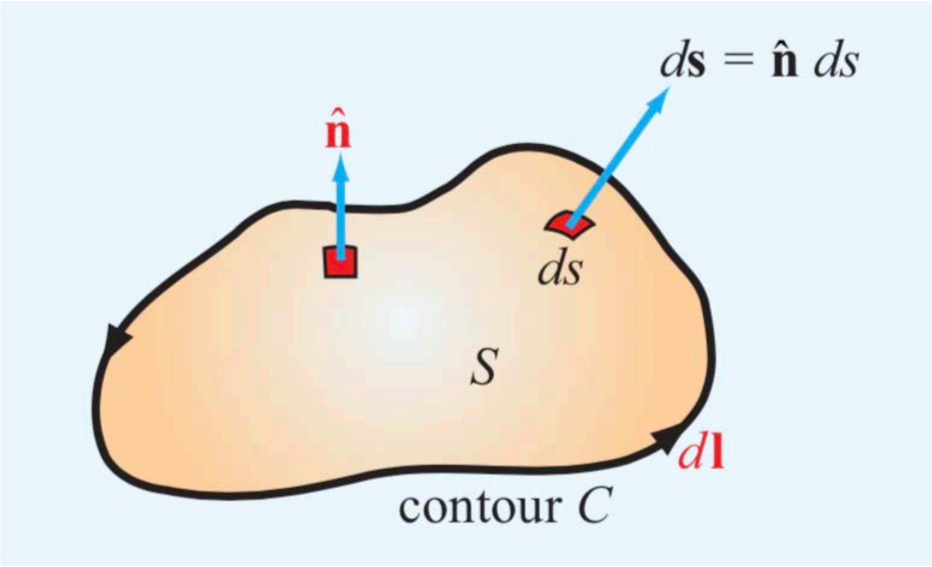
Surface integral of curl of vector over open surface *S*

→ equivalent to line integral of vector along contour, *C*, bounding surface, *S*

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \oint_C \vec{B} \cdot d\vec{l}$$

Surface integral of field curl

Contour line integral of field enclosing surface



Laplacian Operator

Divergence of the gradient of a scalar: $\vec{\nabla} \cdot \left(\vec{\nabla} V\right)$

$$\vec{\nabla} V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} V\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian of scalar: $$\nabla^2 V = \vec{\nabla} \cdot \left(\vec{\nabla} V\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

<u>Laplacian of vector:</u> $$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E}$$

$$\nabla^2 \vec{E} = \hat{x}\underbrace{\nabla^2 E_x}_{\text{Laplacian of vector component}} + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z$$

$$\nabla^2 \vec{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right)$$

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Hold in any material including vacuum

- 1873 James Clark Maxwell obtained from experiments

  by: Coulomb, Gauss, Ampere, Faraday

  Unified theory of electricity and magnetism

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left.\begin{aligned} &\vec{E} = \text{electric field intensity} \\ &\vec{D} = \text{electric field flux density} \end{aligned}\right\} \quad \vec{D} = \epsilon\, \vec{E} \qquad \epsilon = \text{electrical permittivity}$$

$$\left.\begin{aligned} &\vec{H} = \text{magnetic field intensity} \\ &\vec{B} = \text{magnetic flux density} \end{aligned}\right\} \quad \vec{B} = \mu \vec{H} \qquad \mu = \text{magnetic permeability}$$

ρ_V = electric charge density per unit volume

$\vec{J}$ = current density per unit area

Maxwell's Equations

Static → $\frac{\partial}{\partial t} = 0$, ρ_V and $\vec{J}$ are constant

Electrostatics:

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = 0$$

Static → decoupled

Magneto statics:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Electrostatics:

many applications – sensors, displays

Define sources – charge densities, current distributions